J. Fluid Mech. (1976), *vol.* 78, *part* 4, *pp.* 737–747
Printed in Great Britain

Axisymmetric flow of a viscous fluid near the vertex of a body

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(Received 31 March 1976)

Axially symmetric motion of a viscous fluid in a cone is considered on the basis of the Stokes assumption. Near the apex of the cone the solution obtained reveals features quite similar to those of that near a sharp corner in two dimensions, which has been discussed already. An infinite sequence of eddies is induced near the apex for values less than about 80·9° of the semi-angle of the cone, which is measured from the symmetry axis lying in the fluid. The solution found by Pell & Payne for a spindle in a uniform stream offers a good illustration of the general discussion. Special attention is paid to the angle 120° for the spindle as well as the cone. The limiting case of zero angle of the cone corresponds to the flow occurring in a circular cylinder.

1. Introduction

The general features of two-dimensional flow of an incompressible viscous fluid near a sharp corner have been discussed by Moffatt (1964). Later, problems related to such corner eddies were also solved by Schubert (1967) and by the author (1975). A comparable problem in three dimensions is the flow near the apex of a cone. In this paper a solution of the Stokes equations of motion is presented for axisymmetric flow in a space with a conical boundary. The solution near the apex reveals features similar to Moffatt's flow. When the boundary is rigid, the flow sufficiently near the apex consists of an infinite sequence of eddies for certain values of the semi-angle of the cone, which are in general less than about 80·9°. The Stokes assumption is valid sufficiently near the apex of the cone and the behaviour of the fluid there is to some extent independent of the nature of the far field.

The problem of a spindle in a uniform stream has been solved by Pell & Payne (1960). This solution may offer the only example of the above situation other than blunt bodies such as a spheroid. In addition, it is of interest to compare the drag on spindles of various vertex angles, particularly 120°, with the value obtained by Bourot (1974) for the optimum profile with smallest drag, though details on this are beyond the scope of this paper. The solution given by Takagi (1973) for a special torus without a central opening is a limit of the problem of a spindle and is closely related to the limiting case of a cone of zero angle, which gives a possible motion occurring in a circular cylinder.

2. Axisymmetric field with a conical boundary

The axial symmetry of the flow field permits the introduction of a stream function ψ. The equation for ψ in the Stokes regime is known to be

$$L_{-1}^2 \psi = 0. \tag{2.1}$$

In spherical co-ordinates (ρ, θ) with the polar axis as the axis of symmetry, the operator L_k is given by

$$L_k = \frac{\partial^2}{\partial \rho^2} + \frac{1-t^2}{\rho^2}\frac{\partial^2}{\partial t^2} + (k+1)\left(\frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{t}{\rho^2}\frac{\partial}{\partial t}\right), \tag{2.2}$$

where $t = \cos\theta$. It has been shown by Payne (1959) that a solution of (2.1) can be built up using the functions ϕ_k which satisfy the equation

$$L_k \phi_k = 0. \tag{2.3}$$

For odd values of $k = 2m+1$, solutions which are regular on the axis of symmetry can generally be found in the separated form

$$\phi_{2m+1} = A_{2m+1}\rho^{\nu}\frac{d^m}{dt^m}P_{\nu+m}(t) \quad (m = 0, 1, 2, \ldots), \tag{2.4}$$

for any number ν, real or complex, where $P_\nu(t)$ is the Legendre function of the first kind of degree ν.

The general solution to (2.1) of the same separated form as (2.4) is

$$\psi = r^2(\phi_1 + \phi_3) = r^2\rho^\nu\{A_1 P_\nu(t) + A_3 P'_{\nu+1}(t)\}, \tag{2.5}$$

where $r = \rho\sin\theta$ and the prime denotes differentiation with respect to t. Since $P_\nu = P_{-\nu-1}$ and $P'_{\nu+1} = (\nu+1)P_\nu + tP'_\nu$, we may use the alternative representation

$$\psi = r^2[a\rho^\nu + b/\rho^{\nu+1}]\{AP_\nu(t) + BtP'_\nu(t)], \quad \mathscr{R}(\nu) \geqslant -\tfrac{1}{2}, \tag{2.6}$$

where a, b, A and B are arbitrary constants. Here and elsewhere $\mathscr{R}$ denotes 'the real part of'. In the particular cases $\nu = 0$ and $\nu = 1$, the form of ψ degenerates to

$$\psi = (a\rho^2 + b\rho)(A't^2 + B't + C) \tag{2.7}$$

and

$$\psi = (a\rho^3 + b)(A't^3 + B't + C). \tag{2.8}$$

These solutions with separated variables are relevant to problems involving a conical boundary. The boundary is defined by $t_0 = \cos\theta_0$ such that $\theta < \theta_0$ in the fluid. According as $\theta_0 \lesseqgtr 90^\circ$ $(t_0 \gtreqless 0)$, inner or outer flow may be considered. The velocity components of the fluid are

$$v_\rho = \frac{1}{\rho^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{\rho\sin\theta}\frac{\partial\psi}{\partial\rho}. \tag{2.9}$$

The solutions with $\mathscr{R}(\nu) < 2$ in the expression $\psi = \rho^\nu f_\nu(\theta)$ give velocities which are infinite at the origin and tend to zero as $\rho \to \infty$, and so describe the far-field flow when some disturbance is present near the origin. Nevertheless, if we confine ourselves to the situation in which the relative velocity vanishes on the cone surface, these solutions do not produce flows of any practical interest.

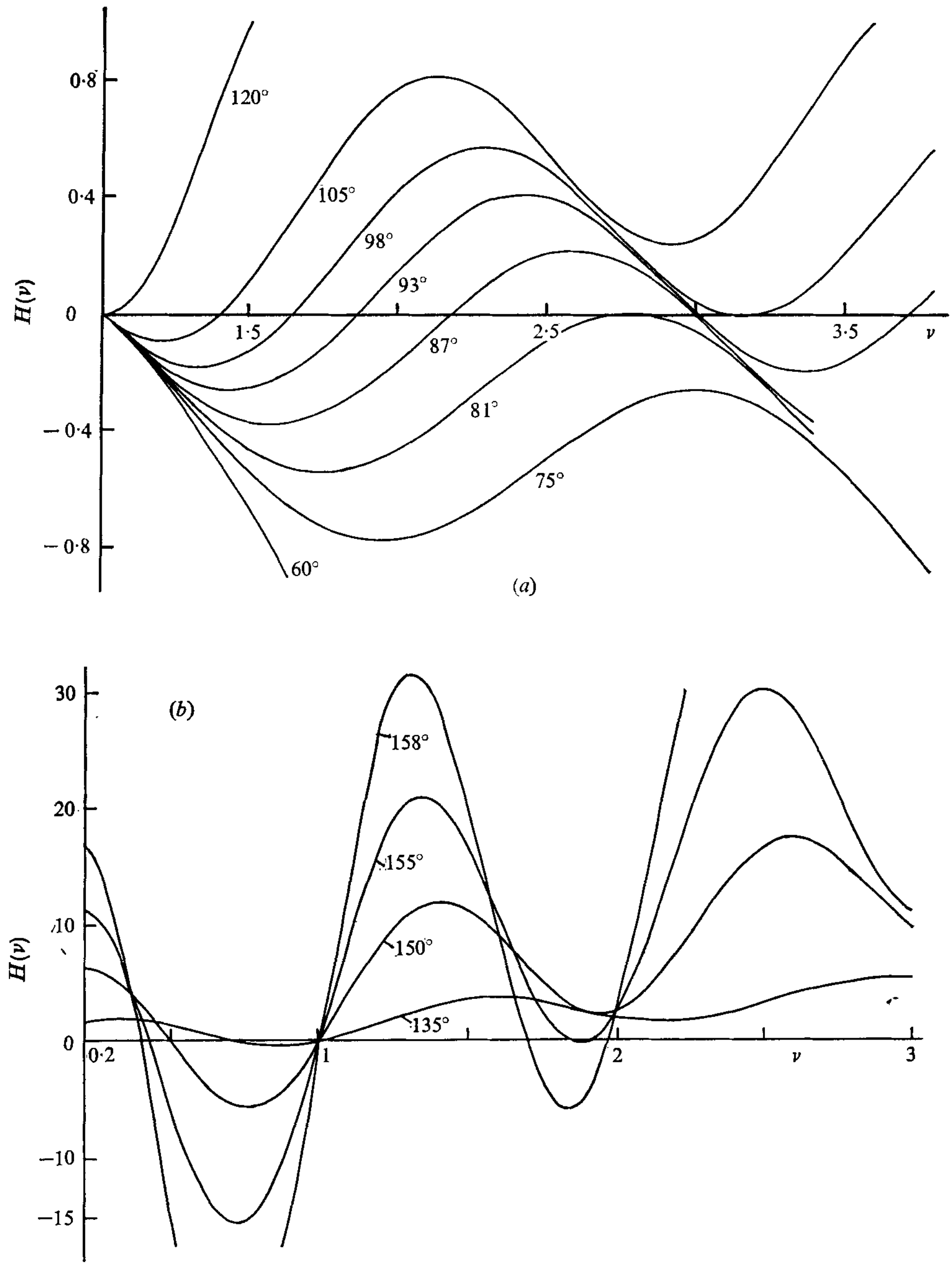

FIGURE 1. Curves of $H(\nu) = P_\nu(t_0)\{t_0 P''_\nu(t_0) + P'_\nu(t_0)\} - t_0\{P'_\nu(t_0)\}^2$ to illustrate the locations of critical angles at which the first or second root of $H(\nu) = 0$ is a double root. (*a*) Neighbourhood of $\theta_0 = 81°–98°$. (*b*) Neighbourhood of $\theta_0 = 155°$.

The only exception under this condition is the flow given by (2.8), i.e.

$$\psi = -A(t-t_0)^2(t+2t_0). \tag{2.10}$$

The velocity corresponding to this is

$$v_\rho = 3A\rho^{-2}(t^2-t_0^2), \quad v_\theta = 0, \tag{2.11}$$

and proves to be a complete solution both for the inner flow ($t > t_0 > 0$) generated by a source of strength $2\pi A(1-t_0)^2(1+2t_0)$ at the apex of a cone and also for the outer flow ($t_0 > t > -t_0$, $t_0 > 0$) generated between two equal-sized cones facing each other with a common apex by a source of strength $-8\pi A t_0^3$ at the common apex.

3. Solution near the apex of a cone

Consider the flow induced near the apex of a cone or a body with a conical front fixed in space by motion at a large distance. It is supposed that in the neighbourhood of the apex, which is at the co-ordinate origin, the stream function can be expanded as

$$\psi = \sum_{n=1} \rho^{\nu_n+2}\{A_n P_{\nu_n}(t) + B_n t P'_{\nu_n}(t)\}\sin^2\theta, \tag{3.1}$$

where the ν_n are ordered such that $0 < \mathscr{R}(\nu_1) < \mathscr{R}(\nu_2) < \ldots$, because the velocity should vanish at the origin. The A_n and B_n are dimensional constants. As discussed by Moffatt for the two-dimensional case, the Reynolds number based on distance from the origin is small for a sufficiently small ρ and the Stokes approximation is valid regardless of the nature of the motion at a large distance. If the boundary conditions

$$\psi = \partial\psi/\partial t = 0 \quad \text{at} \quad t = t_0 \tag{3.2}$$

are imposed, the ν_n must be the roots of the equation

$$P_\nu(t_0)\{t_0 P''_\nu(t_0) + P'_\nu(t_0)\} - t_0\{P'_\nu(t_0)\}^2 = 0, \tag{3.3}$$

or equivalently

$$\frac{d}{dt}\left(\frac{tP'_\nu}{P_\nu}\right) = \frac{d}{dt}\left(\frac{P'_{\nu+1}}{P_\nu}\right) = \frac{d}{dt}\left(\frac{P'_{\nu-1}}{P_\nu}\right) = 0 \quad \text{at} \quad t = t_0. \tag{3.4}$$

In addition to conditions (3.2) it is necessary that

$$\psi = 0 \quad \text{at} \quad t = 1\,(\theta = 0) \tag{3.5}$$

because of the symmetry of the field, but this is satisfied by (3.1) automatically.

The value of ν satisfying (3.3) of course depends on the (given) value of t_0. Although it is difficult to treat (3.3) analytically, it may be observed that (3.3) has an infinite number of roots, of which none is real for θ_0 less than about 81°, only one can be real for θ_0 between 98° and 155° approximately and so forth (figure 1). For the particular case of a plane, i.e. $t_0 = 0$ ($\theta_0 = 90°$), all integral

θ_0 (deg)	ν_1	$\theta_0\nu_1$	$\ln h_1$	$\ln k_1$	ν_2	$\theta_0\nu_2$
0		$4{\cdot}466+1{\cdot}467i$				$7{\cdot}693+1{\cdot}726i$
30	$8{\cdot}064+2{\cdot}614i$	$4{\cdot}22+1{\cdot}37i$	1·20	9·7		
45	$5{\cdot}240+1{\cdot}569i$	$4{\cdot}12+1{\cdot}23i$	2·00	10·5	$9{\cdot}327+1{\cdot}905i$	$7{\cdot}33+1{\cdot}50i$
60	$3{\cdot}841+0{\cdot}9599i$	$4{\cdot}02+1{\cdot}005i$	3·27	12·6	$6{\cdot}892+1{\cdot}219i$	$7{\cdot}22+1{\cdot}28i$
75	$3{\cdot}017+0{\cdot}4369i$	$3{\cdot}95+0{\cdot}571i$	7·19	21·7	$5{\cdot}441+0{\cdot}6689i$	$7{\cdot}12+0{\cdot}876i$
80·8	$2{\cdot}788+0{\cdot}0420i$	$3{\cdot}93+0{\cdot}059i$	74·8	209		
80·9	2·749	3·88			2·819	3·98
90	2	3·14			3	4·71
98	1·642	2·81	$\ln h_2$	$\ln k_2$	3·102	5·31
105	1·397	2·56	10·6	30·9	$2{\cdot}912+0{\cdot}2957i$	$5{\cdot}34+0{\cdot}542i$
120	1	2·09	8·65	21·6	$2{\cdot}497+0{\cdot}3630i$	$5{\cdot}23+0{\cdot}760i$
135	0·7115	1·68	10·3	23·4	$2{\cdot}181+0{\cdot}3062i$	$5{\cdot}14+0{\cdot}721i$
150	0·4894	1·28	20·1	38·9	$1{\cdot}938+0{\cdot}1564i$	$5{\cdot}07+0{\cdot}409i$
155	0·4255	1·15			1·818	4·92
165	0·3071	0·884			1·496	4·23

TABLE 1. Roots of (3.3) with small positive real part for various cone angles. Length and velocity scale factors are added for eddies.

values of ν except $\nu = 1$ satisfy (3.3) and there is no root other than these. This can be seen from the detailed expression

$$\begin{aligned}\psi &= \sum_{n=1} \rho^{2n+2}[B_{2n}\,tP'_{2n}(t) + B_{2n+1}\,\rho\{tP'_{2n+1}(t) - P_{2n+1}(t)\}]\sin^2\theta \\ &= \sum_{n=1} \rho^{2(n+1)}\sin^2\theta\cos^2\theta\{a_n F(1-n, n+\tfrac{3}{2}, \tfrac{3}{2}, t^2) \\ &\qquad + b_n \rho t F(1-n, n+\tfrac{5}{2}, \tfrac{5}{2}, t^2)\},\end{aligned} \tag{3.6}$$

where F denotes the hypergeometric function and a_n and b_n are constants depending on the particular field.

Examining the asymptotic form of ν_n for large n may facilitate the above observation. For sufficiently large $|\nu|$, (3.3) tends asymptotically to

$$\cos(2\nu+1)\,\theta_0 = -\nu\sin 2\theta_0 \quad \text{for} \quad \epsilon \leqslant \theta_0 \leqslant \pi-\epsilon \quad (\epsilon > 0), \tag{3.7}$$

where $\cos\theta_0 = t_0$. Clearly this equation admits no real solution for sufficiently large values of $|\nu|$, provided that $\sin 2\theta_0 \neq 0$. The complex solution of (3.7) may be determined by writing $\nu\theta_0 = \alpha + i\beta$, whereupon (3.7) gives the two real equations

$$\left.\begin{aligned}\cos(2\nu+\theta_0)\cosh 2\beta &= (-\alpha/\theta_0)\sin 2\theta_0, \\ \sin(2\nu+\theta_0)\sinh 2\beta &= (\beta/\theta_0)\sin 2\theta_0.\end{aligned}\right\} \tag{3.8}$$

Solutions of large modulus have the asymptotic forms

$$\alpha_n \sim n\pi + \tfrac{1}{2}(\pi-\theta_0), \quad \beta_n \sim \tfrac{1}{2}\ln\{2n\pi\sin(2\theta_0)/\theta_0\} \quad \text{for} \quad \theta_0 < \tfrac{1}{2}\pi, \tag{3.9}$$

$$\alpha_n \sim n\pi - \tfrac{1}{2}(\pi-\theta'_0), \quad \beta_n \sim \tfrac{1}{2}\ln\{2n\pi\sin(2\theta'_0)/(\pi-\theta'_0)\} \quad \text{for} \quad \theta_0 > \tfrac{1}{2}\pi, \tag{3.10}$$

where $\theta'_0 = \pi - \theta_0$. This ensures that (3.3) has an infinite number of complex roots provided that $\theta_0 \neq \tfrac{1}{2}\pi$. For the particular case $\theta_0 = \tfrac{1}{2}\pi$, (3.7) reduces to

$$\cos\tfrac{1}{2}(2\nu+1)\,\pi = 0. \tag{3.11}$$

Clearly this is satisfied only by integral values of ν_n.

In the special case $\nu = 1$, conditions (3.2) give

$$\psi = A\rho^3(t-t_0)^2(t+2t_0), \tag{3.12}$$

from (2.8). Hence the configuration $t_0 = -\frac{1}{2}$ ($\theta_0 = 120°$) is the only one for which (3.12) also satisfies the condition (3.5), the velocity then being

$$v_\rho = \tfrac{3}{4}A\rho(1-4t^2), \quad v_\theta = \tfrac{3}{4}A\rho(1+2t)^2 \tan\tfrac{1}{2}\theta. \tag{3.13}$$

It is noted that $v_\rho = 0$ everywhere on the line $t = \frac{1}{2}$.

Sufficiently near the origin the first term of (3.1) dominates and asymptotically gives

$$\psi \sim \rho^{\nu_1+2}\{A_1 P_{\nu_1}(t) + B_1 tP'_{\nu_1}(t)\}\sin^2\theta, \tag{3.14}$$

provided that $A_1 \neq 0$ and $B_1 \neq 0$. Thus interest centres chiefly on ν_1, and the values of this quantity were calculated for some cone angles. As θ_0 increases from 0 to π, the value of $\mathscr{R}(\nu_1)$ decreases monotonically. Table 1 shows these values of ν_1 and $\theta_0\nu_1$. If $\nu_1 = \alpha_1 + i\beta_1$ is a complex root, $\bar{\nu}_1 = \alpha_1 - i\beta_1$ is also a root. Thus the solutions occur in conjugate pairs and (3.14) is written as

$$\psi \sim (\rho/\rho_0)^{\alpha_1+2}\sin^2\theta\mathscr{R}[C\exp\{i\beta_1\ln(\rho/\rho_0)\}\{t_0 P'_{\nu_1}(t_0)P_{\nu_1}(t) - P_{\nu_1}(t_0)tP'_{\nu_1}(t)\}], \tag{3.15}$$

where C is a complex constant and ρ_0 is a length scale which is determined by conditions in the given far field. In the same way as in the two-dimensional case, it can be seen that (3.15) implies an infinite sequence of eddies near the origin. A short summary will thus suffice because of the qualitative similarity.

If $\ln(\rho/\rho_0) + V(\theta) = 0$ is a curve on which $\psi = 0$, then ψ is also zero on the curve

$$\ln(\rho/\rho_0) + V(\theta) + n\pi/\beta_1 = 0 \quad (n = 1, 2, \ldots). \tag{3.16}$$

Enclosed within each such curve is an eddy. The distance from the origin to the curve is $\rho_n = \rho_0\exp(-V)\exp(-n\pi/\beta_1)$, so all the eddies are geometrically similar and their dimensions fall off in a geometric progression with ratio $h_1 = \exp(\pi/\beta_1)$. Since the magnitude of the velocity is of order $(\rho_n/\rho_0)^{\alpha_1}$, the ratio of the intensities of consecutive eddies is $k_1 = \exp(\pi\alpha_1/\beta_1)$. The values of h_1 and k_1 are also shown in table 1. Each of these eddies is interpreted as the cross-section of a vortex ring surrounding the axis of symmetry. Since the number of real solutions to (3.3) is finite for $\theta_0 > 81°$ except for $\theta_0 = 90°$, under certain conditions the leading coefficients in (3.1) might vanish and eddies might appear for angles greater than 81°.

It is expected that not only is the flow related to the conical boundary, but in general the solution near the stagnation point of any body may be expanded in a functional form similar to (3.1), the ν_n again being the roots of (3.3) and A_n and B_n depending on the configuration of the body in addition to the nature of the distant field. Axisymmetric flow past a spheroid which is uniform far from the body is well known and gives a proof for the case $t_0 = 0$ ($\theta_0 = 90°$). For example, it is readily seen that near the stagnation point of a sphere

$$\psi \sim \tfrac{3}{4}Ur^2(\rho/b)^2\cos^2\theta. \tag{3.17}$$

For the limiting case of disk, the term proportional to $(\rho/b)^2$ vanishes and asymptotically

$$\psi \sim (2/3\pi)\, Ur^2(\rho/b)^3 \cos^3\theta. \tag{3.18}$$

Here b is the radius of the sphere as well as of the disk, U is the velocity of the undisturbed flow at infinity and $r = \rho\sin\theta$. The solution for a spindle in a uniform stream has been given by Pell & Payne (1960) and seems to be a nice example for angles other than $\theta_0 = 90°$. In this connexion the flow past a spindle will be considered again in the next section.

4. Flow past a spindle or a round mat with a hollow navel

Bipolar co-ordinates (ξ, η) in a plane are defined by

$$z = c\frac{\sinh\xi}{\cosh\xi - \cos\eta}, \quad r = c\frac{\sin\eta}{\cosh\xi - \cos\eta}, \quad c > 0, \tag{4.1}$$

where (z, r) are the cylindrical co-ordinates with the z axis as the symmetry axis. A spindle is described by $\eta = \eta_0$. Although when $\eta_0 < \frac{1}{2}\pi$ the body is a round mat with a hollow navel rather than a spindle, for brevity any body with $\eta = \eta_0$ will be called a spindle. The exterior region is defined by $-\infty < \xi < \infty$, $0 \leqslant \eta < \eta_0$, the point $(\xi = 0, \eta = 0)$ corresponding to infinity. Let the undisturbed flow have a uniform velocity U in the z direction; then the solution for the spindle due to Pell & Payne is given in our notation by

$$\psi = \tfrac{1}{2}Ur^2\left[1 - (\cosh\xi - t)^{\frac{1}{2}}\int_0^\infty \{A(\lambda)\,K_\lambda(t) + B(\lambda)\,tK'_\lambda(t)\}\cos\lambda\xi\, d\lambda\right], \tag{4.2}$$

where

$$A(\lambda) = \frac{2^{\frac{1}{2}}}{\cosh\pi\lambda}\,[\{t_0 K''_\lambda(t_0) + K'_\lambda(t_0)\}\,K_\lambda(-t_0) + t_0 K'_\lambda(t_0)\,K'_\lambda(-t_0)]\,\frac{1}{H(\lambda)}, \tag{4.3a}$$

$$B(\lambda) = -\frac{2^{\frac{1}{2}}}{\cosh\pi\lambda}\,\{K_\lambda(t_0)\,K'_\lambda(-t_0) + K'_\lambda(t_0)\,K_\lambda(-t_0)\}\,\frac{1}{H(\lambda)} \tag{4.3b}$$

and

$$H(\lambda) = K_\lambda(t_0)\,\{t_0 K''_\lambda(t_0) + K'_\lambda(t_0)\} - t_0\{K'_\lambda(t_0)\}^2. \tag{4.3c}$$

In the above representations $t = \cos\eta$, $K_\lambda = P_{-\frac{1}{2}-i\lambda}$, which is a Legendre function of complex degree and is known as a conal function, and $K'_\lambda(-t)$ formally denotes $dK_\lambda(-t)/d(-t)$.

For the sake of the discussion here, we may alternatively write ψ in the form

$$\psi = -\frac{c^2U}{2^{\frac{1}{2}}\pi}\,\frac{\sin^2\eta}{(\cosh\xi - t)^{\frac{3}{2}}}\,e^{-\frac{1}{2}\xi}\int_{-\infty}^{\infty}\frac{G_\nu(t)}{H(\nu)}\,e^{-\nu\xi}\,d\lambda, \tag{4.4}$$

where $\nu = -\frac{1}{2} - i\lambda$, Q_ν is a Legendre function of the second kind and

$$H(\nu) = P_\nu(t_0)\,\{t_0 P''_\nu(t_0) + P'_\nu(t_0)\} - t_0\{P'_\nu(t_0)\}^2, \tag{4.5}$$

$$G_\nu(t) = t_0 P'_\nu(t_0)\frac{d}{dt_0}\,[P_\nu(t_0)\,Q_\nu(t) - Q_\nu(t_0)\,P_\nu(t)] - \frac{d}{dt_0}\,[t_0 P'_\nu(t_0)] \times \{P_\nu(t_0)\,Q_\nu(t) - Q_\nu(t_0)\,P_\nu(t)\} + \frac{tP'_\nu(t)}{1-t_0^2}. \tag{4.6}$$

Here the relations

$$\frac{1}{(\cosh \xi - t)^{\frac{1}{2}}} = 2^{\frac{1}{2}} \int_0^\infty \frac{K_\lambda(-t)}{\cosh \pi\lambda} \cos \lambda\xi \, d\lambda,$$

$$P_\nu(-t) = P_\nu(t) \cos \nu\pi - \frac{2}{\pi} Q_\nu(t) \sin \nu\pi$$

have been used. In particular, when $t_0 = 0$, the expression (4.4) becomes

$$\psi = -\frac{c^2 U}{2^{\frac{3}{2}}} \frac{\sin^2 \eta}{(\cosh \xi - t)^{\frac{3}{2}}} e^{-\frac{1}{2}\xi} \int_{-\infty}^{\infty} \frac{P_\nu(-t) - P_\nu(t) + 2tP'_\nu(t)}{\sin \pi\nu} e^{-\nu\xi} \, d\lambda. \tag{4.7}$$

The integral in (4.4) may be evaluated by the theorem of residues to yield a representation of ψ which is convenient for discussing the flow near the vertices $\xi = \pm\infty$. In the upper half, say, of the complex λ plane, if λ is a root of $H(\lambda) = 0$ then so is $-\lambda$, corresponding to ν and $\bar{\nu}$, respectively. A bar denotes a complex conjugate. Pure imaginary values of λ correspond to real values of ν. From careful examination of $H(\nu)$ and $G_\nu(t)$, it can be seen that $\nu = 0$ is not a pole of the integrand, $\nu = 1$ is a pole only if $t_0 = -\frac{1}{2}$ ($\eta_0 = 120°$) and integers other than these constitute all the poles for $t_0 = 0$ ($\eta_0 = 90°$). In this case (4.7) is rather convenient for giving the solution for a sphere exactly. For non-zero values of t_0, any root, complex or real but not integral, of the equation $H(\nu) = 0$, i.e. (3.3), is a simple pole of the integrand. Thus, using the values of ν_n at these poles, ψ can be represented as a series.

Sufficiently near a vertex of the spindle we have the asymptotic form

$$\psi \sim 2Ug\rho^2 \sin^2 \eta \left(\frac{\rho}{2c}\right)^{\alpha_1} \mathscr{R}\left[\exp(-i\beta_1|\xi|) \frac{G_{\nu_1}(t)}{(dH/d\nu)_{\nu=\nu_1}}\right] \tag{4.8}$$

because $\rho = 2ce^{-|\xi|}$ approximately, where ρ is the distance from the origin, $\nu_1 = \alpha_1 + i\beta_1$ is the root of (3.3) with smallest positive real part, and $g = \frac{1}{2}$ for real roots ($\beta_1 = 0$) and $g = 1$ for complex roots. For the spindle with $\eta_0 = 120°$, (4.8) is reduced to

$$\psi \sim \tfrac{4}{45}(U/c)\rho^3(1-t)(1+2t)^2 \tag{4.9}$$

because $\nu_1 = 1$. As may be seen from (3.15), eddies are observed for complex roots, which are inevitable for η_0 less than about 81°.

The force exerted on the spindle has been calculated by Stasiw *et al.* (1974) for $\eta_0 > 90°$ in 5° steps. Their numerical values seem, unfortunately, to be incorrect for a reason which will be given later, and so the force was recalculated using (Payne & Pell 1960)

$$\begin{aligned} D &= 8\pi\mu \lim_{\rho\to\infty} \left[\frac{\rho}{r^2}\left(\psi - \frac{U}{2} r^2\right)\right] \\ &= 4\pi\mu c U \int_0^\infty \left[\frac{d}{dt}\{(1-t^2) K''_\lambda(t)\} K_\lambda(-t) - (1-t^2) K''_\lambda(t) \frac{d}{dt} K_\lambda(-t)\right]_{t=t_0} \\ &\qquad \times \frac{d\lambda}{H(\lambda)\cosh \pi\lambda}, \end{aligned} \tag{4.10}$$

where μ is the coefficient of viscosity of the fluid. Some numerical values of $D/6\pi\mu bU$ are listed in table 2, in which a is half the largest thickness of the

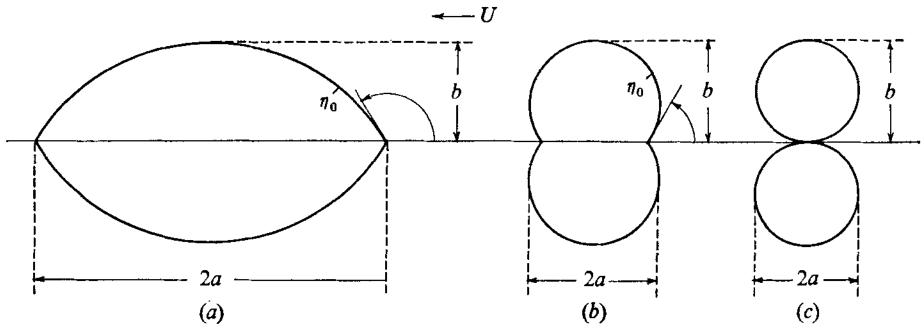

FIGURE 2. Diagram of the typical bodies involved. (*a*) Spindle. (*b*) Round mat. (*c*) Closed torus.

		$D/6\pi\mu bU$		C_f	
η_0 (deg)	a/b	Spindle	Spheroid	Spindle	Spheroid
150	3·732	1·473	1·5468	1·016	0·9972
120	1·732	1·101	1·1492	0·960	0·9569
90	1	1	1	1	1
60	0·667	0·958	0·9352	1·053	1·0705
30	0·536	0·938	0·9115	1·095	1·1222
0	0·5	0·935*	0·9053	1·115	1·1406

TABLE 2. Drag coefficients for spindles of various thickness. The starred value is for a closed torus and is due to Takagi.

spindle measured along the mainstream and b is the semi-axis perpendicular to it, so that $a = c$ for $\eta_0 \geqslant 90°$, $a = c/\sin\eta_0$ for $\eta_0 \leqslant 90°$ and $b = c\,(1+\cos\eta_0)/\sin\eta_0$. The results for spheroids are compared with those with the same value of a/b.

The volume of a spindle may be shown to be

$$\frac{2\pi b^3}{(1+\cos\eta_0)^3}\{(\pi-\eta_0)\cos\eta_0+(1-\tfrac{1}{3}\sin^2\eta_0)\sin\eta_0\}. \tag{4.11}$$

We also take a particular interest in the ratio C_f of the drag to that on the sphere of equal volume, in connexion with the fact that the profile of given volume having the smallest drag in a uniform flow should have conical front and rear ends of angle 120° (Pironneau 1973). This coefficient C_f for a spindle of angle 120° is indeed greater than the coefficient for the optimum profile, which was given by Bourot as $C_f = 0{\cdot}95425$, but is still fairly small (table 2). The values of the drag coefficient given by Stasiw *et al.* disagree with those in table 2, being too large, and also are at variance with the conclusion of Pironneau (their values of C_f increase monotonically with increasing η_0).

5. Flow occurring in a circular cylinder

The limiting form of (3.3) for $\theta_0 \to 0$ can be obtained using the relation

$$\lim_{\nu\to\infty} \nu^n P_\nu^{-n}(\cos x/\nu) = J_n(x),$$

where P_ν^{-n} is an associated Legendre function and J_n is a Bessel function. The limit is taken such that $\theta_0 \nu = \zeta$, a finite value, as $\theta_0 \to 0$; then in the limit the equation for ζ is found to be

$$\zeta\{J_0^2(\zeta)+J_1^2(\zeta)\}-2J_0(\zeta)J_1(\zeta) = 0, \quad \zeta \neq 0. \tag{5.1}$$

Equation (2.3) is rewritten as

$$L_k \phi_k = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial r^2}+\frac{k}{r}\frac{\partial}{\partial r}\right)\phi_k = 0, \tag{5.2}$$

referred to the cylindrical co-ordinates (z, r) with the z axis as the symmetry axis. In order to have a stream function appropriate to the limiting problem we may again start from solutions to (5.2) with k odd of the form

$$\phi_{2m+1} = e^{-\gamma|z|}J_m(\gamma r)/(\gamma r)^m \quad (m = 0, 1, 2, \ldots). \tag{5.3}$$

Then instead of equation (3.1) for ψ we obtain

$$\psi = \sum_{n=1} \exp(-\gamma_n|z|)\, r^2\left\{A_n J_0(\gamma_n r)+B_n\frac{J_1(\gamma_n r)}{\gamma_n r}\right\}, \tag{5.4}$$

which is supposed to give a possible motion occurring in a circular cylinder. The boundary conditions (3.2) at $r = r_0$, the radius of the cylinder, just yield (5.1) for determining $\gamma_n r_0 = \zeta$. When $|\zeta|$ is sufficiently large, (5.1) is reduced to

$$\cos 2\zeta = -2\zeta; \tag{5.5}$$

then for large n, ζ_n will have the asymptotic form

$$\zeta_n \sim \tfrac{1}{2}\{(2n+1)\pi \pm i\ln(4n\pi)\}. \tag{5.6}$$

The solution (5.4) was also obtained by Fitz-Gerald (1972), who calculated the first ten roots of (5.1), though for another purpose.

The roots $\gamma_1 r_0 = p_1 \pm iq_1$ with smallest positive real part (table 1), appearing in conjugate pairs, give the asymptotic solution

$$\psi \sim 2r^2\exp(-p_1|z|/r_0)\mathscr{R}[\exp(-iq_1|z|/r_0)\{A_1 J_0(\gamma_1 r)+B_1 J_1(\gamma_1 r)/\gamma_1 r\}], \tag{5.7}$$

at a large distance from a disturbance which is present near the origin. Clearly this solution implies a sequence of eddies all of the same size of diminishing strength. Consequently the flow field is made up of a file of vortex rings surrounding the central axis. The distance between the centres of adjacent rings, with mutually reciprocal rotational velocities, is

$$\pi r_0/q_1 = 2{\cdot}142 r_0. \tag{5.8}$$

The solution for a closed torus without a central opening moving with velocity U in the direction of the axis of symmetry is comparable with that near a cusped corner in two dimensions, treated by Schubert, and is also closely related to the above situation. The stream function was obtained by Takagi (1973) in terms of tangent-sphere co-ordinates (ξ, η, ϕ) related to the cylindrical system (z, r, ϕ) by

$$z = c\eta/(\xi^2+\eta^2), \quad r = c\xi/(\xi^2+\eta^2) \quad (0 \leqslant \xi < \infty, -\infty < \eta < \infty, 0 \leqslant \phi < 2\pi), \quad c > 0,$$

and in our notation is given by

$$\psi = \frac{c^2 U}{\pi} \frac{\xi}{(\xi^2+\eta^2)^{\frac{3}{2}}} \int_0^\infty [\xi\{\Delta_1(l\xi_0)\, K_0(l\xi) - \Delta_2(l\xi_0)\, I_0(l\xi)\} + \xi_0\, I_1(l\xi)] \cos l\eta \, \frac{dl}{\Delta_1(l\xi_0)},$$

with $\quad \Delta_1(l\xi_0) = l\xi_0\{I_0^2(l\xi_0) - I_1^2(l\xi_0)\} - 2I_0(l\xi_0)\, I_1(l\xi_0),$

$$\Delta_2(l\xi_0) = l\xi_0\{I_0(l\xi_0)\, K_0(l\xi_0) + I_1(l\xi_0)\, K_1(l\xi_0)\} - 2I_1(l\xi_0)\, K_0(l\xi_0),$$

where I_n and K_n are the modified Bessel functions of the first and second kinds of order n, and the surface of the torus is given by $\xi = \xi_0$.

In order to have a suitable form near the origin $\eta = \infty$, ψ may again be evaluated by contour integratron in the upper half, say, of the complex l plane. Taking $l\xi_0 = i\zeta$, we have

$$\psi = \frac{U}{2\pi} r\rho \int_{-\infty}^{\infty} \frac{F_\zeta(\xi)}{\Delta_1(\zeta)} \exp(-\zeta\eta/\xi_0)\, dl, \tag{5.9}$$

where $\rho = (z^2+r^2)^{\frac{1}{2}}$ and

$$\Delta_1(\zeta) = i[\zeta\{J_0^2(\zeta) + J_1^2(\zeta)\} - 2J_0(\zeta)\, J_1(\zeta)].$$

The explicit form of $F_\zeta(\xi)$ used for contraction will be evident without display. If l is a pole of the integrand, so is $-\bar{l}$, corresponding to $\bar{\zeta}$ and ζ, respectively. The point $\zeta = 0$ is not a pole and all the poles are found from (5.1). Consequently (5.9) can be expanded like (5.7) by the same procedure as in the case of the spindle.

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